Activity 13

The poor cartographer—*Graph coloring*

Age group	Early elementary and up.
Abilities assumed	Coloring in.
Time	30 minutes or more.
Size of group	Suitable for individuals to the whole class.

Focus

Problem solving.

Logical reasoning.

Algorithmic procedures and complexity.

Communication of insights.

Summary

Many optimization problems involve situations where certain events cannot occur at the same time, or where certain members of a set of objects cannot be adjacent. For example, anyone who has tried to time-table classes or meetings will have encountered the problem of satisfying the constraints on all the people involved. Many of these difficulties are crystallized in the map coloring problem, in which colors must be chosen for countries on a map in a way that makes bordering countries different colors. This activity is about that problem.

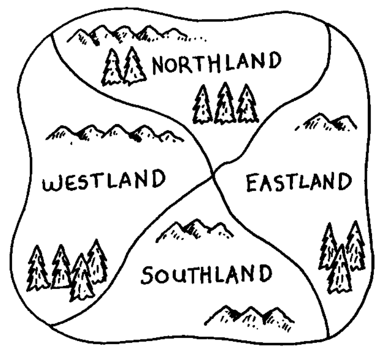


Figure 13.1: A sample map to be colored

Technical terms

Graph coloring; exponential time algorithms; heuristics.

Materials

For each child you will need:

a copy of the blackline masters on pages 138 to 141,

movable small colored markers (e.g. counters or poker chips), and

four crayons of different colors (or colored pencils, felt tips etc.)

You will also need:

a blackboard or similar writing surface.

What to do

This activity revolves around a story in which the children have been asked to help out a cartographer, or map-maker, who is coloring in the countries on a map. It doesn't matter which color a country is, so long as it's different to all bordering countries. For example, Figure 13.1 shows four countries. If we color Northland red, then Westland and Eastland cannot be red, since their border with Northland would be hard to see. We could color Westland green, and it is also acceptable to color Eastland green because it does not share a border with Westland. (If two countries meet only at a single point, they do not count as sharing a border and hence can be made the same color.) Southland can be colored red, and we end up needing only two colors for the map.

In our story, the cartographer is poor and can't afford many crayons, so the idea is to use as few colors as possible.

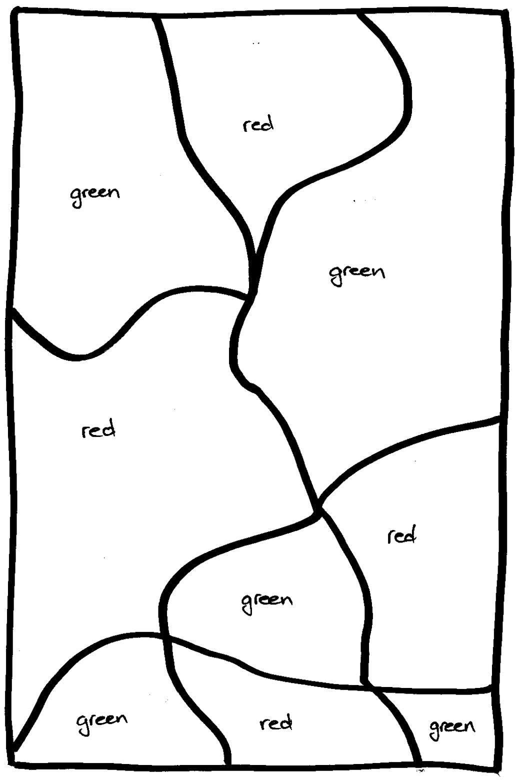


Figure 13.2: Solution for coloring the map on page 138 using just two colors.

1. Describe the problem that the children will be working on, demonstrating the coloring process on a blackboard.

2. Give out the blackline master on page 138. This map can be colored correctly using only two colors. Although restricting the number of colors to just two might sound particularly challenging, the task is quite simple compared with maps that require more colors because there is very little choice about what color each country can be.

   Have the children try to color the map in with only two colors. In the process they may discover the "has-to-be" rule: once one country is colored in, any bordering country has to be the opposite color. This rule is applied repeatedly until all countries are colored in. It is best if the children can discover this rule for themselves, rather than being told it, as it will give them a better insight into the process. Figure 13.2 shows the only possible solution for the map on page 138 (of course, the choice of colors is up to the child, but only two different ones are required).

   The children may also discover that it is better to use place-holders, such as colored counters, instead of coloring the countries straight away, since this makes it easier for

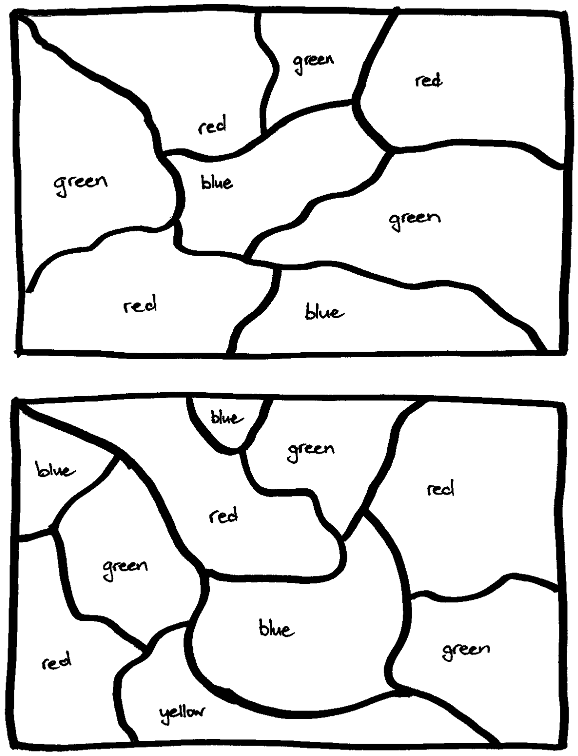


Figure 13.3: Solutions for coloring the maps on page 139 using just three and four colors respectively.

them to change their mind.

3. As children complete each exercise they can be given the next sheet to try. The map at the top of page 139 can be colored correctly using three colors, while the one at the bottom requires four. Possible solutions are shown in Figure 13.3. The map on page 140 is a simpler three-color map, with a possible solution shown in Figure 13.4.

   For older children, ask them to explain how they know that they have found the minimum number of colors. For example, at least three colors are required for the map in Figure 13.4 because the map includes a group of three countries (the largest three), each of which has borders with the other two.

4. If a child finishes all the sheets early, ask them to try to devise a map that requires five different colors. It has been proved that *any* map can be colored with only four colors, so this task will keep them occupied for some time! In our experience children will quickly find maps that they believe require five colors, but of course it is always possible to find a

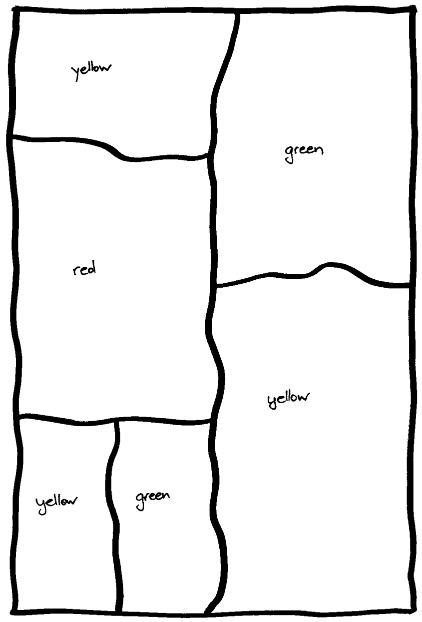


Figure 13.4: Solution for coloring the map on page 140 using just three colors.

four-color solution to their maps.

Variations and extensions

There is a simple way to construct maps that require only two colors, like the one on page 141 (solution in Figure 13.5). This map was drawn by overlaying closed curves (lines whose beginning joins up with their end). You can draw any number of these curves, of any shape, on top of each other, and you will always end up with a map that can be colored with two colors. Children can experiment with creating this type of map.

Four colors are always enough to color a map drawn on a sheet of paper or on a sphere (that is, a globe). One might wonder (as scientists are paid to do) how many colors are needed for maps drawn on weirder surfaces, such as the torus (the shape of a donut). In this case, one might need five colors, and five is always enough. Children might like to experiment with this.

There are many other entertaining variations on the map-coloring problem that lead off into directions where much is currently unknown. For example, if I am coloring a map on a sheet of paper by myself, then I know that if I work cleverly, four colors will be enough. But suppose that instead of working alone I am working with an incompetent (or even adversarial) partner. Assume that I work cleverly, while my partner only works “legally” as we take turns coloring countries on the map. How many crayons need to be on the table in order for me in my cleverness to be able to make up for my partner’s legal but not very bright (or even subversive) moves? Presently we know that 33 crayons will always be enough, but we don’t know that this

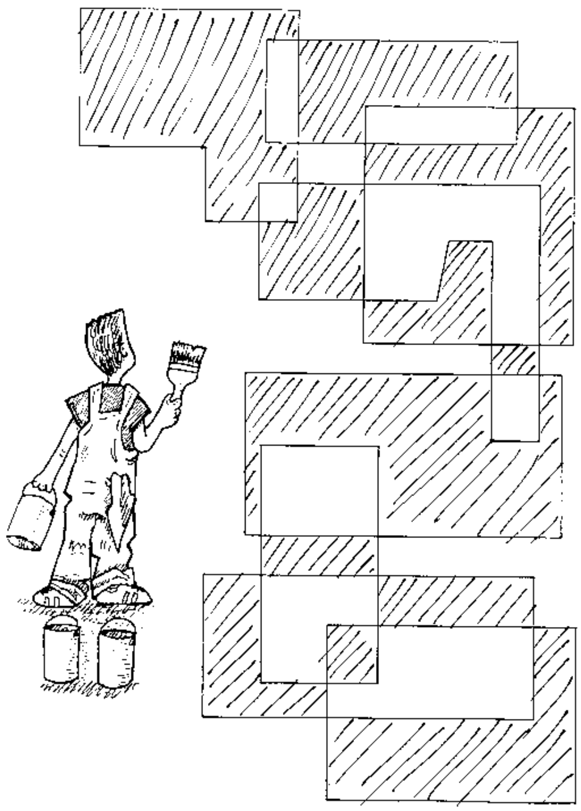


Figure 13.5: Solution for coloring the map on page 141 using just two colors. The colors are shown in this figure as shaded and white.

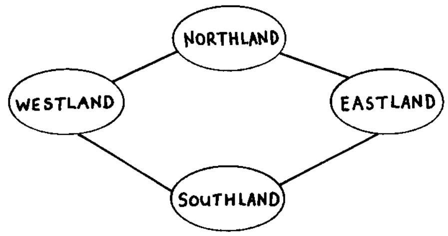


Figure 13.6: An example of a graph

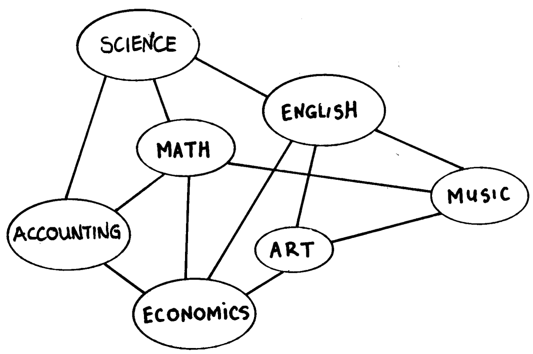


Figure 13.7: Another graph

many is ever actually required. (Experts conjecture that fewer than 10 colors are sufficient.) Children might enjoy acting out this situation, which is rather like a two-person game.

In a variation know as empire coloring, we start with two different maps on two sheets of paper having the same number of countries. Each country on one of the maps (say, the Earth) is paired with exactly one country on the other map (which might be colonies on the Moon). In addition to the usual coloring requirement of different colors for countries that share a border (for both maps) we add the requirement that each Earth country must be colored the same as its colony on the Moon. How many colors do we need for this problem? The answer is currently unknown.

What's it all about?

The map coloring problem that we have explored in this activity is essentially to find the minimum number of colors—two, three, or four—that are necessary to color a particular map. The conjecture that any map can be colored using only four colors was formulated in 1852, but it was not proved until 1976. Computer science is full of unsolved problems, and knowing that the four-color theorem was proved after more than 120 years of attention from researchers is an encouragement to those working on other problems whose solution has eluded them for decades.

Map coloring belongs to a general class of problems known as "graph coloring." In computer science, a graph is an abstract representation of relationships. An example is shown in Figure 13.6.

As mentioned in Activity 9 on the Muddy City, the term *graph* is used in a different sense in mathematics to mean a chart displaying numerical data, such as a bar graph, but the graphs that computer scientists use are not related to these. In computer science, graphs are drawn using circles or large dots, technically called "nodes," to denote objects, with lines between them to indicate some sort of relationship between the objects. The graph in Figure 13.6 happens to represent the map of Figure 13.1. The nodes represent the countries, and a line between two nodes indicates that those two countries share a common border. On the graph, the coloring rule is that no connected nodes should be allocated the same color. Unlike a map, there is no limit to the number of colors that a general graph may require, because many different constraints may be drawn in as connecting lines, whereas the two-dimensional nature of maps restricts the possible arrangements. The "graph coloring problem" is to find the minimum number of colors that are needed for a particular graph.

Figure 13.7 shows another graph. Here the nodes correspond to subjects in a school. A line between two subjects indicates that at least one student is taking both subjects, and so they should not be timetabled for the same period. Using this representation, the problem of finding a workable timetable using the minimum number of periods is equivalent to the coloring problem, where the different colors correspond to different periods. Graph coloring algorithms are of great interest in computer science, and are used for many real-world problems, although they are probably never used to color in maps!—our poor cartographer is just a fiction.

There are literally thousands of other problems based on graphs. Some are described elsewhere in this book, such as the minimal spanning tree of Activity 9 and the dominating sets of Activity 14. Graphs are a very general way of representing data and can be used to represent all sorts of situations, such as routes available for travel between cities, connections between atoms in a molecule, paths that messages can take through a computer network, connections between components on a printed circuit board, and relationships between the tasks required to carry out a large project. For this reason, problems involving graph representations have long fascinated computer scientists.

Many of these problems are very difficult—not difficult conceptually, but difficult because they take a long time to solve. For example, to determine the most efficient solution for a graph coloring problem of moderate size—such as finding the best way to timetable a school with thirty teachers and 800 students—would take years, even centuries, for a computer using the best known algorithm. The problem would be irrelevant by the time the solution was found—and that's assuming the computer doesn't break down or wear out before it finishes! Such problems are only solved in practice because we are content to work with sub-optimal, but still very good, solutions. If we were to insist on being able to guarantee that the solution found was the very best one, the problem would be completely intractable.

The amount of computer time needed to solve coloring problems increases exponentially with the size of the graph. Consider the map coloring problem. It can be solved by trying out all possible ways to color the map. We know that at most four colors are required, so we need to evaluate every combination of assigning the four colors to the countries. If there are n countries, there are 4^n combinations. This number grows very rapidly: every country that is added multiplies the number of combinations by four, and hence quadruples the solution time. Even if a computer were invented that could solve the problem for, say, fifty countries in just one hour, adding one more country would require four hours, and we would only need to add ten more countries to make the computer take over a year to find the solution. This kind of problem won't go away just because we keep inventing faster and faster computers!

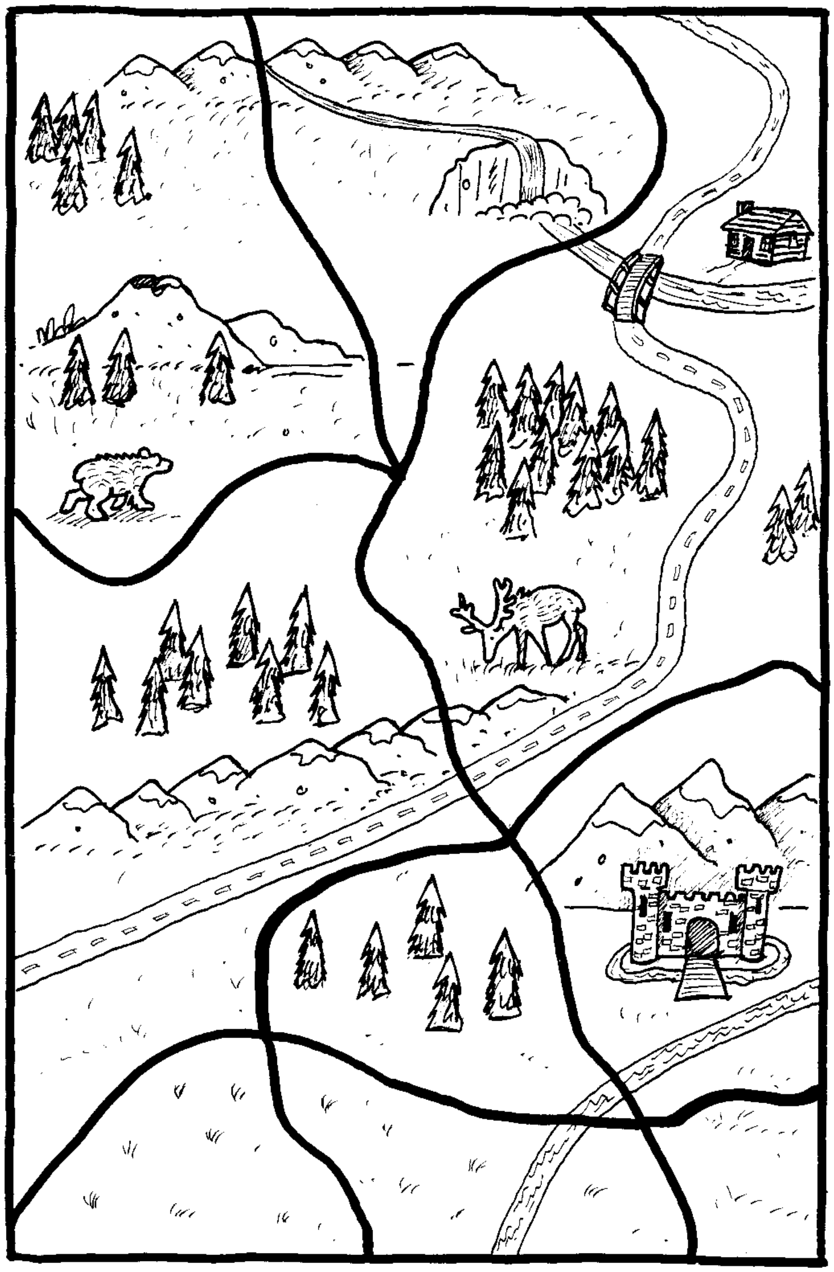
Graph coloring is a good example of a problem whose solution time grows exponentially. For very simple instances of the problem, such as the small maps used in this activity, it is quite easy to find the optimal solution, but as soon as the number of countries increases beyond about ten, the problem becomes very difficult to do by hand, and with a hundred or more countries, even a computer would take many years to try out all the possible ways of coloring the map in order to choose the optimal one.

Many real-life problems are like this, but must be solved anyway. Applied computer scientists use methods that give good, but not perfect, answers. These *heuristic* techniques are often very close to optimal, very fast to compute, and give answers that are close enough for all practical purposes. Schools can tolerate using one more classroom than would be needed if the timetable were perfect, and perhaps the poor cartographer could afford an extra color even though it is not strictly necessary.

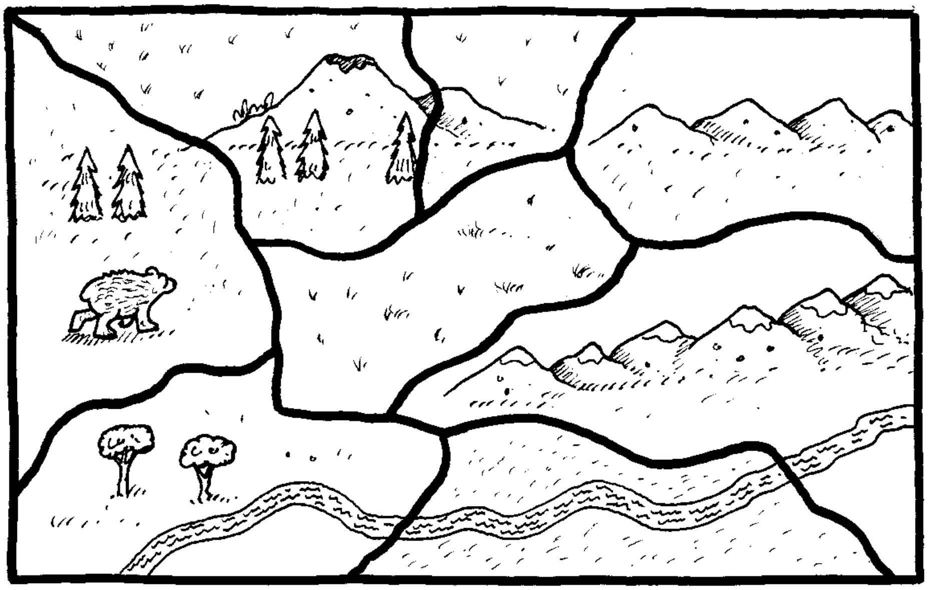
No-one has proved that there isn't an efficient way to solve this sort of problem on conventional computers, but neither has anyone proved that there is, and computer scientists are sceptical that an efficient method will ever be found. We will learn more about this kind of problem in the next two activities.

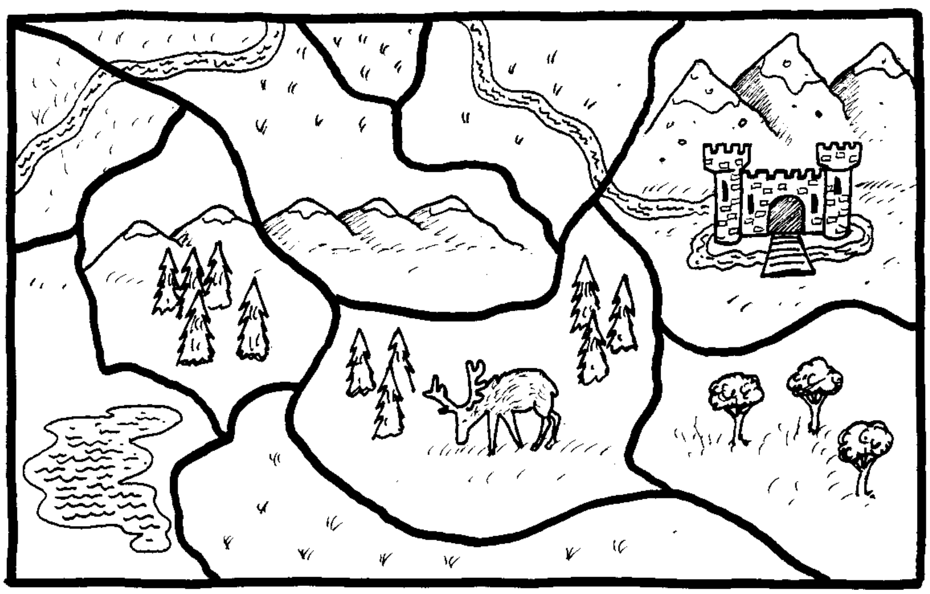
Further reading

Harel discusses the four-color theorem, including its history, in *Algorithmics*. More aspects of the map-coloring problem are discussed in *This is MEGA-Mathematics!* by Casey and Fellows. More information about graph coloring can be found in Beineke and Wilson's book *Selected Topics in Graph Theory*, and Garey and Johnson's book *Computers and Intractability*.

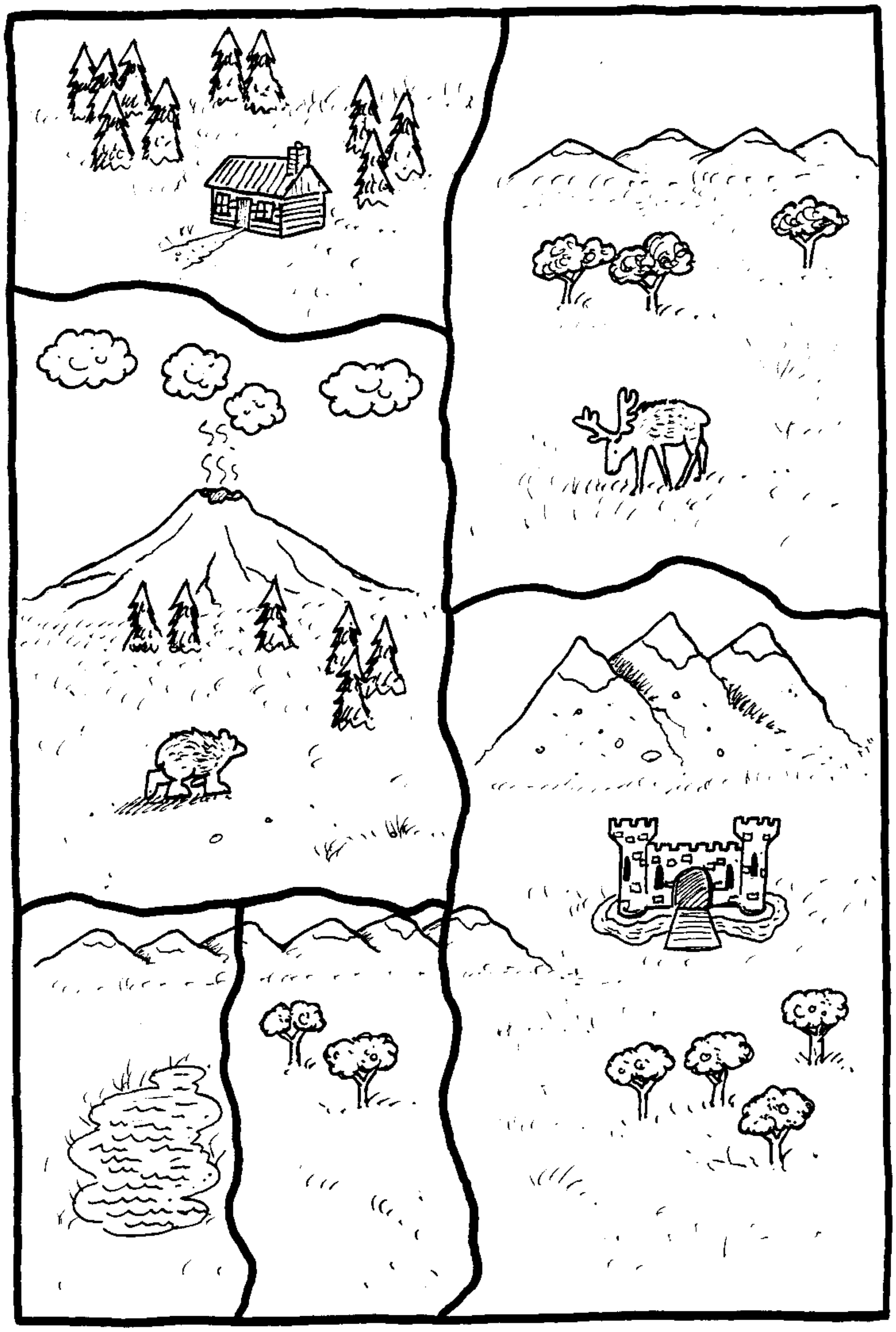


Instructions: *Color in the countries on this map with as few colors as possible, but make sure that no two bordering countries are the same color.*





Instructions: *Color in the countries on these maps with as few colors as possible, but make sure that no two bordering countries are the same color.*



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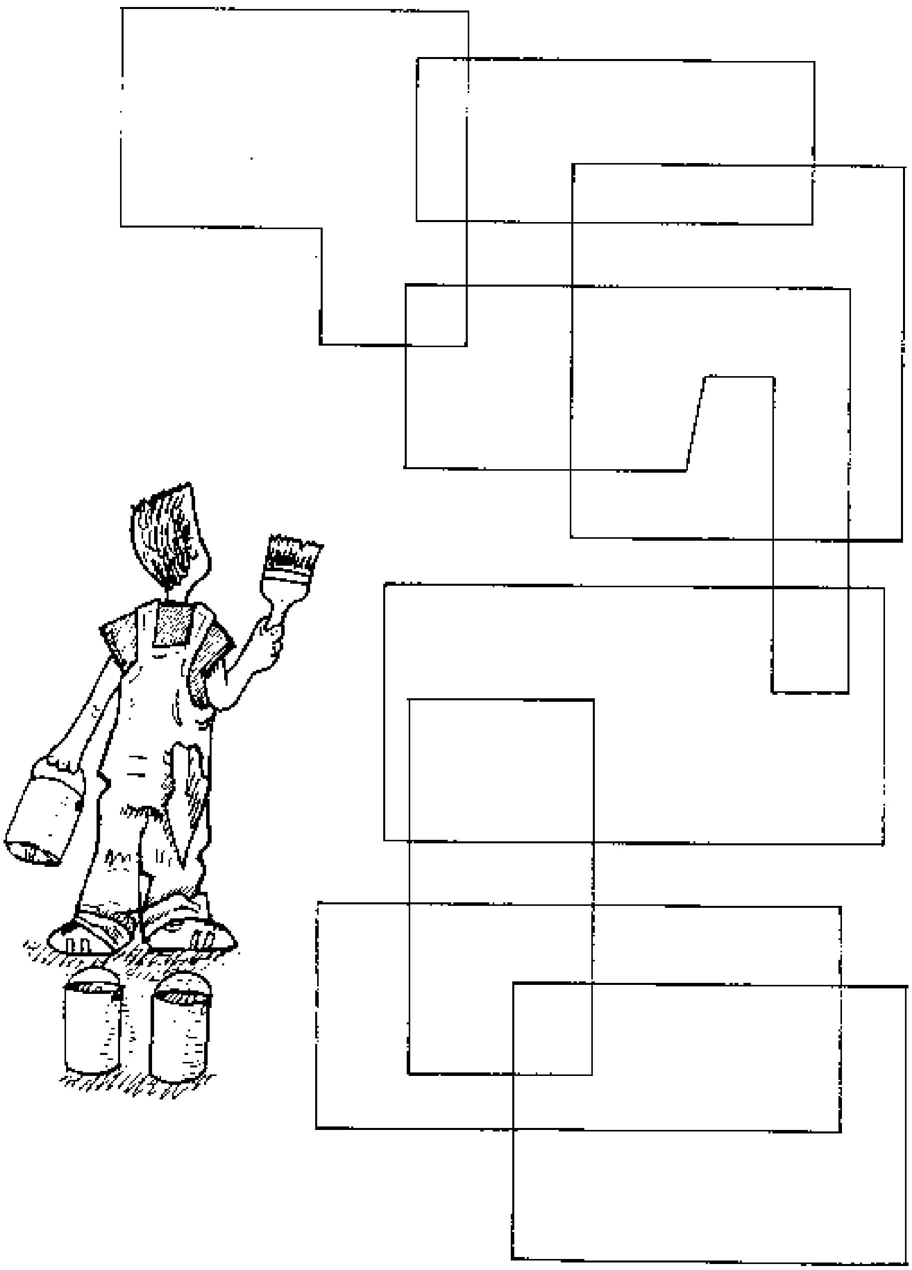

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